INTERNATIONAL BACCALAUREATE

MATHEMATICS

Higher Level

Thursday 11 May 1995 (morning)

Paper 2 | 2 hours 30 minutes

This examination paper consists of 2 sections, Section A and Section B.

Section A consists of 4 questions.

Section B consists of 4 questions.

The maximum mark for Section A is 80.

The maximum mark for each question in Section B is 40.

The maximum mark for this paper is 120.

This examination paper consists of 10 pages.

INSTRUCTIONS TO CANDIDATES

DO NOT open this examination paper until instructed to do so.

Answer all FOUR questions from Section A and ONE question from Section B.

Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures as appropriate.

EXAMINATION MATERIALS

Required/Essential:

- IB Statistical Tables
- Millimetre square graph paper
- Electronic calculator
- Ruler and compasses

Allowed/Optional:

A simple translating dictionary for candidates not working in their own language

FORMULAE

Trigonometrical identities:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha - \cos\beta = 2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}$$

$$\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \cos^2\theta - \sin^2\theta$$

$$\text{If } \tan\frac{\theta}{2} = t \text{ then } \sin\theta = \frac{2t}{1+t^2} \text{ and } \cos\theta = \frac{1-t^2}{1+t^2}$$

Integration by parts:

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

Standard integrals:

$$\int \frac{\mathrm{d}x}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + c$$

$$\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + c \qquad (|x| < a)$$

Statistics: If $(x_1, x_2, \ldots, x_n)$ occur with frequencies $(f_1, f_2, \ldots, f_n)$ then the mean m and standard deviation s are given by

$$m = \frac{\Sigma f_i x_i}{\Sigma f_i} \qquad s = \sqrt{\frac{\Sigma f_i (x_i - m)^2}{\Sigma f_i}}, \qquad i = 1, 2, \ldots, n$$

Binomial distribution:

$$p_x = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \ldots, n$$

A correct answer with **no** *indication of the method used will normally receive* **no** *marks. You are therefore advised to show your working.*

SECTION A

Answer all FOUR questions from this section.

1. *[Maximum mark: 18]*

(i) The fifth, seventh and twelfth terms of the arithmetic sequence a_1, a_2, a_3, ... are consecutive terms of a geometric sequence. Find the common ratio of the geometric sequence. *[9 marks]*

(ii) The sum of the first k positive integers can be written as

$$1 + 2 + 3 + \ldots . + k = \frac{k(k+1)}{2}.$$

Given $n \in \mathbb{N}$ find, in terms of n, the sum of the integers between 1 and $15n$ inclusive which are not divisible by 3 or by 5. Simplify your answer as much as possible. *[9 marks]*

2. *[Maximum mark: 20]*

(i) (a) Show that the lines

$$x = 3 + 4t \qquad x = -1 + 12s$$
$$y = 4 + t \qquad y = 7 + 6s$$
$$z = 1 \qquad z = 5 + 3s$$

intersect, and find the coordinates of the point of intersection.

(b) Find the equation of the plane that contains these two lines. *[11 marks]*

(ii) An aeroplane approaches the end of a runway from the west with an angle of descent 24°. A tower is 18 metres high and it is situated 135 metres north and 250 metres west of the end of the runway.

(a) Write down the position vector of the top of the tower relative to an origin at the end of the runway, and a vector in the direction of the line of descent of the aeroplane.

(b) Hence, or otherwise, calculate how close, to the nearest metre, the aeroplane comes to a warning light that is at the top of the tower.

[Note: The distance between a point with position vector $\vec{a}$ and a line through the origin and in the direction $\vec{u}$ is given by $\dfrac{|\vec{a} \times \vec{u}|}{|\vec{u}|}$.] *[9 marks]*

3. *[Maximum mark: 24]*

(i) (a) Find the tangent to the curve of

$$f(x) = \frac{x}{(7x^2 + 5)}$$

at the point when $x = 1$.

(b) Find the area under the curve of $f(x)$ between $x = 0$ and $x = \sqrt{3}$.

(c) By setting $s = \tan x$, or otherwise, find

$$\int_0^{\frac{\pi}{3}} \frac{\tan x \, dx}{7\sin^2 x + 5\cos^2 x}.$$

[14 marks]

(ii) (a) Show that the function $g(x) = \log_{10} x$ can be written as

$$g(x) = \frac{1}{\ln 10} \ln x.$$

(b) Differentiate the function $h(x) = \dfrac{\log_{10} x}{x}$.

Given that the only stationary point of $h(x)$ is a maximum deduce, **without** calculating any numerical values, that $e^{\pi} > \pi^{e}$.

[10 marks]

4. *[Maximum mark: 18]*

(i) Eggs are packed in boxes of twelve. Records in a particular store indicate that 77.9% of the boxes have no broken eggs, 19.4% have one broken egg, 2.6% have two broken eggs and 0.1% have three broken eggs. The probability that there are more than three broken eggs in a box is zero.

An egg is selected at random from a box and is found to be broken. What is the probability that it is the only broken egg in the box? *[6 marks]*

(ii) A multiple choice test consists of 50 questions, and for each question the candidate has a choice of three answers. Using the normal approximation to the binomial distribution, find the probability that a candidate who guesses every answer gets between 10 and 16 answers correct, inclusive. *[6 marks]*

(iii) The probability density function of a continuous random variable X is

$$f(x) = \begin{cases} kxe^{-\lambda x}, & x \geqslant 0, \\ 0, & \text{elsewhere,} \end{cases}$$

where k is a constant and λ is a positive constant.

Express k in terms of λ. *[6 marks]*

SECTION B

Answer ONE question from this section.

Abstract Algebra

5. *[Maximum mark: 40]*

   (i) Write down the axioms that must be satisfied if the set S, together with the binary operation $*$, is to form a group.

   Show that matrix multiplication is a binary operation on the set of all 2×2 matrices of the form

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},$$

   where $n \in \mathbb{Z}$, and that this set is a group under the operation of matrix multiplication. *[14 marks]*

   (ii) (a) Let A be the set $\{1, 2, 3\}$ and P be the set of all permutations of the elements of A.

   Two of the elements of P are thus

$$p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \qquad p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

   Write down the remaining elements, denoting the elements of P by p_i.

   (b) Given that P is a group under the operation of composition, write down the group table.

   (c) Use the group table to determine a subgroup of order 3. *[12 marks]*

   (iii) By considering the group tables of

   1. the set $\{1, i, -1, -i\}$ with respect to multiplication, where $i = \sqrt{-1}$, and

   2. the set $\{m_1, m_2, m_3, m_4\}$ with respect to matrix multiplication, where

$$m_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad m_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

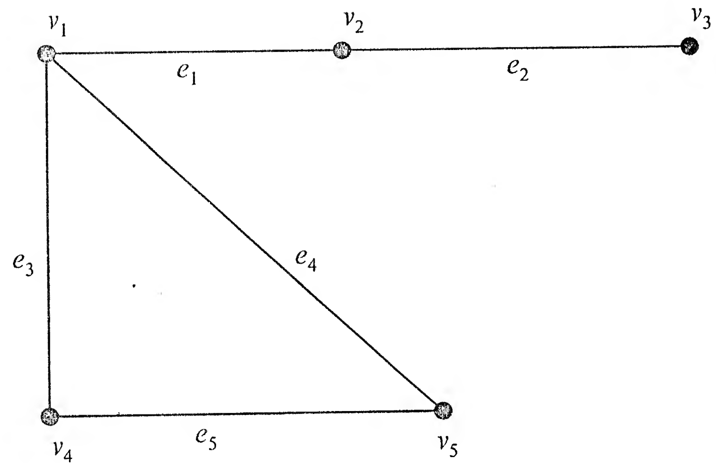
$$m_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad m_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

   show that the two groups are isomorphic. Explain clearly what this means. *[14 marks]*

Graphs and Trees

6. *[Maximum mark: 40]*

(i) (a) Write down the **adjacency matrix** A and the **incidence matrix** B of the graph below.



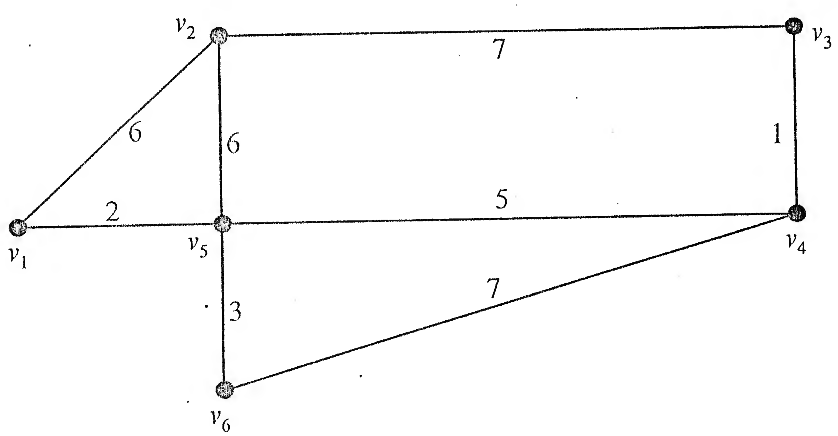
(b) Explain clearly the significance of the entries in the matrix A^k. Verify your answer for $k = 2$ in the above example, and list all pairs of vertices involved. *[14 marks]*

(ii) (a) Define the terms **Eulerian** and **Hamiltonian** as applied to a connected simple graph G (that is, there are no loops or multiple edges and there is a path between any two distinct vertices of G).

State a condition that is both necessary and sufficient for G to be Eulerian, and a condition that is sufficient for G to be Hamiltonian.

(b) Give an example of a simple connected graph with six vertices which is Hamiltonian but not Eulerian. Explain clearly the reasons why, and write down a sequence of edges that satisfies the Hamiltonian requirement. *[15 marks]*

(iii) Find a minimal spanning tree, and its numeric value, for the graph below.



[11 marks]

Turn over

Statistics

7. *[Maximum mark: 40]*

(a) The blood pressures of a sample of 30 adult males were recorded in mm. The sample mean and standard deviation are 130.33 and 10.54 respectively, the latter being calculated from $\frac{1}{29}\sum_{i=1}^{30}(x_i - \overline{x})^2$, where $\overline{x}$ is the sample mean.

Explain how to calculate a 95% confidence interval for the mean blood pressure in the population from which the sample was drawn. Distributional results used in the derivation of your answer should be clearly stated.

Calculate the end points of the 95% confidence interval given the above sample values. Would the results lead to the acceptance of a population mean of 135 when testing at the 5% significance level against a two-sided alternative hypothesis? *[14 marks]*

(b) In an independent sample of two hundred from the same population, thirty seven individuals claim that they exercise regularly. Stating clearly all the assumptions made and the distributional results used, obtain an approximate 95% confidence interval for the proportion of the population who exercise regularly. Would this lead you to reject the claim that 25% of the population exercise regularly? *[13 marks]*

(c) Members of a further sample of three hundred were classified by the attributes

$E \equiv$ Exercise regularly
$E^* \equiv$ Do not exercise regularly
$M \equiv$ Manual worker
$M^* \equiv$ Non-manual worker

and the resulting contingency table was:

	E	E^*
M	22	118
M^*	38	122

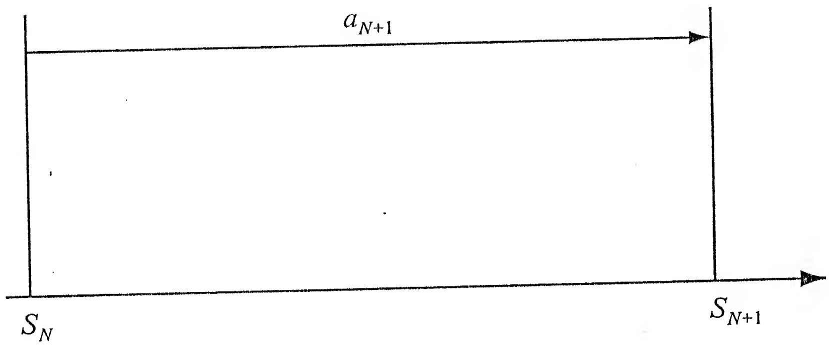
Investigate whether or not there is any association between the two attributes. Once again a test at the 5% significance level will be satisfactory. *[13 marks]*

Analysis and Approximation

8. *[Maximum mark: 40]*

(i) (a) The series $\sum_{n=1}^{\infty} a_n$ is an alternating series such that $|a_{n+1}| < |a_n|$ for all n and $a_n \to 0$ as $n \to \infty$. Let $S_N = \sum_{n=1}^{N} a_n$.

Copy the diagram below and, by inserting a_{N+2}, S_{N+2}, a_{N+3}, use it to demonstrate geometrically, that the series converges **and** that $|S - S_N| < |a_{N+1}|$ for all values of N, where S is the limit of the series.



[7 marks]

(b) Approximate the sum $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!}$ to within an error of less than $\frac{1}{2} \times 10^{-5}$. *[9 marks]*

(c) Use the Maclaurin series

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots$$

to approximate $\int_0^{0.5} \sin(x^2)\,\mathrm{d}x$ to three decimal places.

[You may assume that the operations of integration and summation can be interchanged.] *[6 marks]*

(This question continues on page 10)

(*Question 8 continued*)

(ii) Write down Simpson's rule for approximating the integral $\int_a^b f(x)\,\mathrm{d}x$ using $2n+1$ points.

Denoting this approximation by $S(f)$, the error is known to be

$$\int_a^b f(x)\,\mathrm{d}x - S(f) = K(b-a)h^4 f^{(\mathrm{iv})}(c)$$

where $a \leqslant c \leqslant b$, $h = \dfrac{b-a}{2n}$ and K is a constant.

Given that K is the same for **any** function $f(x)$, **any** finite values of a and b ($a <$ b) and **any** value of n, derive the value of this constant.

Calculate the value of n to ensure that the approximation to $\int_1^2 \frac{1}{x}\,\mathrm{d}x$ has an error less than $\frac{1}{2} \times 10^{-6}$.

[18 marks]